

Presentations for (singular) partition monoids: a new approach

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Abstract

We give new, short proofs of the presentations for the partition monoid and its singular ideal originally given in the author's 2011 papers in *J. Alg.* and *I.J.A.C.*

Keywords: Partition monoid, Singular ideal, Presentations.

MSC: 20M05; 20M20.

1 Introduction

Partition monoids arise in the construction of partition algebras [16, 21] as twisted semigroup algebras [14, 25]. Although the partition algebras (and other related diagram algebras) were originally introduced in the context of theoretical physics and representation theory, their applications have been many and varied; see for example the recent surveys [17, 22]. In particular, diagram monoids have played an increasingly important role in semigroup theory in the last two decades; see for example [1–3, 9, 12, 18], and especially [10] for an extensive list of references. This semigroup approach was first employed by Wilcox in the context of cellular algebras [25], and was also instrumental in providing the first full proof [6] of a presentation (by generators and relations) for the partition algebras; the presentation was first stated in [14]. The method in [6] was to first obtain a presentation for the partition monoid (stated in Theorem 2.2 below), and then apply a general result on twisted semigroup algebras (which was also proved in [6]).

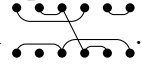
Taking classical work of Howie on singular transformation semigroups [15] as inspiration, the singular ideals of partition monoids were investigated in [7], the main result being a presentation by generators and relations (stated in Theorem 2.1 below); the generating set consists of idempotents and is of minimal possible size. (See also [20], where a similar presentation was given for the singular part of the Brauer monoid.) The article [7] unlocked some intriguing combinatorial properties of the partition monoids, and paved the way for several further studies, most notably [10], which concerns the question of (minimal) idempotent generation of arbitrary ideals in several natural families of diagram monoids. The proofs of Theorems 2.1 and 2.2 given in [6, 7] relied on several previous results [4, 5, 11, 24] to obtain initial (highly complex) presentations, which were subsequently reduced (using lengthy sequences of Tietze transformations); the articles [6, 7] have a combined length of 58 pages.

The purpose of the current article is to give shorter, more direct, and conceptually simpler proofs of Theorems 2.1 and 2.2. Apart from quoting two results (Theorem 2.1 of [8] and Theorem 2 of [11]),

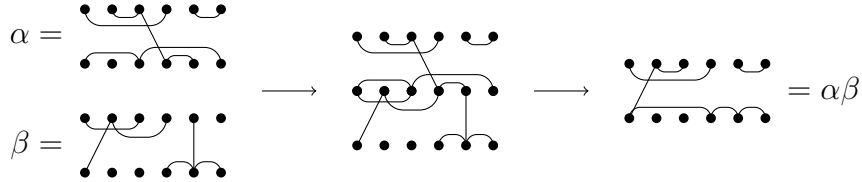
the current article is entirely self-contained: we give the required definitions and state the main results in Section 2, and then prove them in Sections 3 and 4. We believe that the new, efficient techniques for working with presentations are of independent interest, as may be the normal forms given in Proposition 3.14.

2 Preliminaries and statement of the main results

Fix an integer $n \geq 2$ (everything is trivial if $n \leq 1$), and write $\mathbf{n} = \{1, \dots, n\}$ and $\mathbf{n}' = \{1', \dots, n'\}$. The *partition monoid of degree n* , denoted \mathcal{P}_n , consists of all set partitions of $\mathbf{n} \cup \mathbf{n}'$, under a product described below. Such a partition $\alpha \in \mathcal{P}_n$ may be represented graphically. We draw vertices $1, \dots, n$ on an upper row (increasing from left to right) with $1', \dots, n'$ directly below, and add edges so that connected components of the graph correspond to the blocks of α . For example, the partition $\alpha = \{\{1, 4\}, \{2, 3, 4', 5'\}, \{5, 6\}, \{1', 3', 6'\}, \{2'\}\} \in \mathcal{P}_6$ is represented by the graph



The product of two partitions $\alpha, \beta \in \mathcal{P}_n$ is calculated as follows. We first stack (graphs representing) α and β so that lower vertices $1', \dots, n'$ of α are identified with upper vertices $1, \dots, n$ of β . The connected components of this graph are then constructed, and we finally delete the middle row; the resulting graph is the product $\alpha\beta \in \mathcal{P}_n$. Here is an example calculation with $\alpha, \beta \in \mathcal{P}_6$:



The operation is associative, so \mathcal{P}_n is a semigroup: in fact, a monoid, with identity $1 = \begin{smallmatrix} \bullet & \bullet & \dots & \bullet \\ \vdots & \vdots & & \vdots \\ \bullet & \bullet & \dots & \bullet \end{smallmatrix}$.

A block of a partition $\alpha \in \mathcal{P}_n$ is called a *transversal block* if it has non-trivial intersection with both \mathbf{n} and \mathbf{n}' ; otherwise, it is a *non-transversal block* (which may be an *upper* or *lower* non-transversal block, with obvious meanings). The *rank* of α , denoted $\text{rank}(\alpha)$, is the number of transversal blocks of α . For example, $\alpha \in \mathcal{P}_6$ as above has transversal block $\{2, 3, 4', 5'\}$ (so $\text{rank}(\alpha) = 1$), upper non-transversal blocks $\{1, 4\}$, $\{5, 6\}$, and lower non-transversal blocks $\{1', 3', 6'\}$, $\{2'\}$.

For $\alpha \in \mathcal{P}_n$ and $x \in \mathbf{n} \cup \mathbf{n}'$, we write $[x]_\alpha$ for the block of α containing x . We then define the *domain* and *codomain* of α to be the sets

$$\text{dom}(\alpha) = \{x \in \mathbf{n} : [x]_\alpha \cap \mathbf{n}' \neq \emptyset\} \quad \text{and} \quad \text{codom}(\alpha) = \{x \in \mathbf{n} : [x']_\alpha \cap \mathbf{n} \neq \emptyset\}.$$

We also define the *kernel* and *cokernel* of α to be the equivalences

$$\ker(\alpha) = \{(x, y) \in \mathbf{n} \times \mathbf{n} : [x]_\alpha = [y]_\alpha\} \quad \text{and} \quad \text{coker}(\alpha) = \{(x, y) \in \mathbf{n} \times \mathbf{n} : [x']_\alpha = [y']_\alpha\}.$$

For example, with $\alpha \in \mathcal{P}_6$ as as above,

$$\text{dom}(\alpha) = \{2, 3\}, \quad \text{codom}(\alpha) = \{4, 5\}, \quad \ker(\alpha) = (1, 4 \mid 2, 3 \mid 5, 6), \quad \text{coker}(\alpha) = (1, 3, 6 \mid 2 \mid 4, 5),$$

using an obvious notation for equivalences.

It is immediate from the definitions that the following hold for all $\alpha, \beta \in \mathcal{P}_n$:

$$\text{dom}(\alpha\beta) \subseteq \text{dom}(\alpha), \quad \ker(\alpha\beta) \supseteq \ker(\alpha), \quad \text{codom}(\alpha\beta) \subseteq \text{codom}(\beta), \quad \text{coker}(\alpha\beta) \supseteq \text{coker}(\beta).$$

Also, $\text{rank}(\alpha\beta\gamma) \leq \text{rank}(\beta)$ for all $\alpha, \beta, \gamma \in \mathcal{P}_n$.

If $\alpha \in \mathcal{P}_n$, we will write

$$\alpha = \left(\begin{array}{c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ \hline B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \right)$$

to indicate that α has transversal blocks $A_i \cup B'_i$ (for $1 \leq i \leq r$), upper non-transversal blocks C_j (for $1 \leq j \leq p$), and lower non-transversal blocks D'_k (for $1 \leq k \leq q$). (For $A \subseteq \mathbf{n}$, we write $A' = \{a' : a \in A\}$.) If α (as above) has no non-transversal blocks, we will write $\alpha = \left(\begin{array}{c|c|c} A_1 & \cdots & A_r \\ \hline B_1 & \cdots & B_r \end{array} \right)$. Such a partition is also known as a *block bijection*, and the set \mathcal{J}_n of all such block bijections is (isomorphic to) the *dual symmetric inverse monoid of degree n* ; see [6, 13]. If each block of α (as above) intersects \mathbf{n} in at most one point and also intersects \mathbf{n}' in at most one point, we will write $\alpha = \left[\begin{array}{c|c|c} a_1 & \cdots & a_r \\ \hline b_1 & \cdots & b_r \end{array} \right]$ to indicate that α has transversal blocks $\{a_i, b'_i\}$ (for $1 \leq i \leq r$), all other blocks being singletons. Such a partition is also known as a *partial permutation*, and the set \mathcal{I}_n of all such partial permutations is (isomorphic to) the *symmetric inverse monoid*; see [7, 19]. As usual, with $\alpha \in \mathcal{I}_n$ as above, we will write $a_i\alpha = b_i$ for each i . The group of units of \mathcal{P}_n is (isomorphic to) the *symmetric group* $\mathcal{S}_n = \mathcal{J}_n \cap \mathcal{I}_n$. A permutation π of \mathbf{n} is identified with the partition $\left(\begin{array}{c|c|c} 1 & \cdots & n \\ \hline 1\pi & \cdots & n\pi \end{array} \right) = \left[\begin{array}{c|c|c} 1 & \cdots & n \\ \hline 1\pi & \cdots & n\pi \end{array} \right]$. The set $\mathcal{P}_n \setminus \mathcal{S}_n = \{\alpha \in \mathcal{P}_n : \text{rank}(\alpha) < n\}$ of non-invertible (i.e., *singular*) partitions is a subsemigroup (indeed, an ideal) of \mathcal{P}_n .

In order to state the above-mentioned presentations for $\mathcal{P}_n \setminus \mathcal{S}_n$ and \mathcal{P}_n from [6, 7], we first fix the notation we will be using for presentations. Let X be an alphabet, and denote by X^+ (resp., X^*) the free semigroup (resp., free monoid) on X . If $R \subseteq X^+ \times X^+$ (resp., $R \subseteq X^* \times X^*$), we denote by $R^\#$ the congruence on X^+ (resp., X^*) generated by R . We say a semigroup (resp., monoid) S has *semigroup* (resp., *monoid*) *presentation* $\langle X : R \rangle$ if $S \cong X^+/R^\#$ (resp., $S \cong X^*/R^\#$) or, equivalently, if there is an epimorphism $X^+ \rightarrow S$ (resp., $X^* \rightarrow S$) with kernel $R^\#$. If ϕ is such an epimorphism, we say S has *presentation* $\langle X : R \rangle$ *via* ϕ . A relation $(w_1, w_2) \in R$ will usually be displayed as an equation: $w_1 = w_2$. We will always be careful to specify whether a given presentation is a semigroup or monoid presentation. We denote the *empty word* (over any alphabet) by 1, so $X^* = X^+ \cup \{1\}$. If A is a subset of a semigroup S , then $\langle A \rangle$ always denotes the *subsemigroup* generated by A .

For $1 \leq r \leq n$ and $1 \leq i < j \leq n$, define partitions

$$\bar{e}_r = \begin{array}{c} 1 \cdots r \cdots n \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \end{array} \quad \text{and} \quad \bar{t}_{ij} = \bar{t}_{ji} = \begin{array}{c} 1 \cdots i \cdots j \cdots n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}.$$

Consider alphabets $E = \{e_r : 1 \leq r \leq n\}$ and $T = \{t_{ij} : 1 \leq i < j \leq n\}$, and define a (semigroup) homomorphism

$$\phi : (E \cup T)^+ \rightarrow \mathcal{P}_n \setminus \mathcal{S}_n$$

by $e_r\phi = \bar{e}_r$ and $t_{ij}\phi = \bar{t}_{ij}$ for each $1 \leq r \leq n$ and $1 \leq i < j \leq n$. We will use symmetric notation when referring to the letters from T , so we write $t_{ij} = t_{ji}$ for all $1 \leq i < j \leq n$. Consider the relations

$$e_i^2 = e_i \quad \text{for all } i \quad (\text{R1})$$

$$e_i e_j = e_j e_i \quad \text{for distinct } i, j \quad (\text{R2})$$

$$t_{ij}^2 = t_{ij} \quad \text{for all } i, j \quad (\text{R3})$$

$$t_{ij} t_{kl} = t_{kl} t_{ij} \quad \text{for all } i, j, k, l \quad (\text{R4})$$

$$t_{ij} t_{jk} = t_{jk} t_{ki} \quad \text{for distinct } i, j, k \quad (\text{R5})$$

$$t_{ij} e_k = e_k t_{ij} \quad \text{if } k \notin \{i, j\} \quad (\text{R6})$$

$$t_{ij} e_k t_{ij} = t_{ij} \quad \text{if } k \in \{i, j\} \quad (\text{R7})$$

$$e_k t_{ij} e_k = e_k \quad \text{if } k \in \{i, j\} \quad (\text{R8})$$

$$e_k t_{ki} e_i t_{ij} e_j t_{jk} e_k = e_k t_{kj} e_j t_{ji} e_i t_{ik} e_k \quad \text{for distinct } i, j, k \quad (\text{R9})$$

$$e_k t_{ki} e_i t_{ij} e_j t_{jl} e_l t_{lk} e_k = e_k t_{kl} e_l t_{li} e_i t_{ij} e_j t_{jk} e_k \quad \text{for distinct } i, j, k, l. \quad (\text{R10})$$

Here is the first of our main results; it originally appeared in [7, Theorem 46].

Theorem 2.1. *The semigroup $\mathcal{P}_n \setminus \mathcal{S}_n$ has semigroup presentation $\langle E \cup T : (R1-R10) \rangle$ via ϕ .*

In order to state the second main result, define partitions

$$\bar{e} = \bar{e}_1 = \begin{array}{c} 1 \quad \dots \quad n \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array}, \quad \bar{t} = \bar{t}_1 = \begin{array}{c} 1 \quad \dots \quad n \\ \bullet \quad \bullet \quad \dots \quad \bullet \\ \bullet \quad \bullet \quad \dots \quad \bullet \end{array}, \quad \bar{s}_i \Phi = \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad n \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \end{array} \quad \text{for } 1 \leq i \leq n-1.$$

Consider the alphabet $S = \{s_i : 1 \leq i \leq n-1\}$, and define a (monoid) homomorphism

$$\Phi : (S \cup \{e, t\})^* \rightarrow \mathcal{P}_n$$

by $e\Phi = \bar{e}$, $t\Phi = \bar{t}$ and $s_i\Phi = \bar{s}_i$ for each i . Consider the relations

$$s_i^2 = 1 \quad \text{for all } i \quad (R11)$$

$$s_i s_j = s_j s_i \quad \text{if } |i - j| > 1 \quad (R12)$$

$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i - j| = 1 \quad (R13)$$

$$e^2 = e = ete \quad (R14)$$

$$t^2 = t = tet = ts_1 = s_1 t \quad (R15)$$

$$es_i = s_i e \quad \text{if } i \geq 2 \quad (R16)$$

$$ts_i = s_i t \quad \text{if } i \geq 3 \quad (R17)$$

$$s_1 e s_1 e = e s_1 e s_1 = e s_1 e \quad (R18)$$

$$ts_2 t s_2 = s_2 t s_2 t \quad (R19)$$

$$t(s_2 s_3 s_1 s_2) t (s_2 s_3 s_1 s_2) = (s_2 s_3 s_1 s_2) t (s_2 s_3 s_1 s_2) t \quad (R20)$$

$$t(s_2 s_1 e s_1 s_2) = (s_2 s_1 e s_1 s_2) t. \quad (R21)$$

Here is our second main result; this originally appeared in [6, Theorem 32].

Theorem 2.2. *The monoid \mathcal{P}_n has monoid presentation $\langle S \cup \{e, t\} : (R11-R21) \rangle$ via Φ .*

We conclude this section by stating the presentation for $\mathcal{I}_n \setminus \mathcal{S}_n$ from [8]. (Recall that \mathcal{I}_n was defined above.) With this in mind, for $i, j \in \mathbf{n}$ with $i \neq j$, define $\bar{f}_{ij} = \bar{e}_i \bar{t}_{ij} \bar{e}_j$ (using symmetric notation for $\bar{t}_{ij} = \bar{t}_{ji}$). Note that $\bar{f}_{ij} \neq \bar{f}_{ji}$; rather,

$$\bar{f}_{ij} = \begin{cases} \begin{array}{c} 1 \quad \dots \quad i \quad \dots \quad j \quad \dots \quad n \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \end{array} & \text{if } i < j \\ \begin{array}{c} 1 \quad \dots \quad j \quad \dots \quad i \quad \dots \quad n \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \end{array} & \text{if } j < i. \end{cases}$$

Define an alphabet $F = \{f_{ij} : i, j \in \mathbf{n}, i \neq j\}$, and consider the relations

$$f_{ij} f_{ji} f_{ij} = f_{ij} \quad \text{for distinct } i, j \quad (F1)$$

$$f_{ij}^3 = f_{ij}^2 = f_{ji}^2 \quad \text{for distinct } i, j \quad (F2)$$

$$f_{ij} f_{kl} = f_{kl} f_{ij} \quad \text{for distinct } i, j, k, l \quad (F3)$$

$$f_{ij} f_{ji} = f_{ik} f_{ki} \quad \text{for distinct } i, j, k \quad (F4)$$

$$f_{ij} f_{ik} = f_{jk} f_{ij} = f_{ik} f_{jk} \quad \text{for distinct } i, j, k \quad (F5)$$

$$f_{ki} f_{ij} f_{jk} = f_{kj} f_{ji} f_{ik} \quad \text{for distinct } i, j, k \quad (F6)$$

$$f_{ki} f_{ij} f_{jk} f_{kl} = f_{kl} f_{li} f_{ij} f_{jl} \quad \text{for distinct } i, j, k, l. \quad (F7)$$

The next result is [8, Theorem 2.1].

Theorem 2.3. *The semigroup $\mathcal{I}_n \setminus \mathcal{S}_n$ has semigroup presentation $\langle F : (F1-F7) \rangle$ via $f_{ij} \mapsto \bar{f}_{ij}$. \square*

3 Presentation for $\mathcal{P}_n \setminus \mathcal{S}_n$

There are two components to the proof of Theorem 2.1: that the map $\phi : (E \cup T)^+ \rightarrow \mathcal{P}_n \setminus \mathcal{S}_n$ is an epimorphism; and that $\ker \phi$ is generated by relations (R1–R10). Proposition 3.1 accomplishes the first task, and the remainder of the section is devoted to the second.

We write \mathfrak{Eq}_n for the set of all equivalence relations on \mathbf{n} , which we regard as a semilattice (monoid of commuting idempotents) under \vee ; the *join*, $\varepsilon \vee \eta$, of two equivalences $\varepsilon, \eta \in \mathfrak{Eq}_n$ is defined to be the smallest equivalence containing $\varepsilon \cup \eta$ (see [11, 13]).

For a subset $A \subseteq \mathbf{n}$ with $\mathbf{n} \setminus A = \{i_1, \dots, i_k\}$, we write $\bar{t}_A = \left(\begin{smallmatrix} A & i_1 & \dots & i_k \\ A & i_1 & \dots & i_k \end{smallmatrix} \right)$. Note that $\bar{t}_A = 1$ if $|A| \leq 1$. Note that if $i, j \in \mathbf{n}$ with $i \neq j$, then $\bar{t}_{\{i,j\}} = \bar{t}_{ij}$ in the notation of the previous section. Note that if $A = \{a_1, a_2, \dots, a_r\}$ with $r = |A| \geq 2$, then $\bar{t}_A = \bar{t}_{a_1 a_2} \bar{t}_{a_2 a_3} \cdots \bar{t}_{a_{r-1} a_r} = \bar{t}_{a_1 a_2} \bar{t}_{a_1 a_3} \cdots \bar{t}_{a_1 a_r}$. For an equivalence $\varepsilon \in \mathfrak{Eq}_n$ with equivalence classes A_1, \dots, A_k , we write $\bar{t}_\varepsilon = \left(\begin{smallmatrix} A_1 & \dots & A_k \\ A_1 & \dots & A_k \end{smallmatrix} \right) = \bar{t}_{A_1} \cdots \bar{t}_{A_k}$. In fact, the set $\{\bar{t}_\varepsilon : \varepsilon \in \mathfrak{Eq}_n\}$ is equal to $E(\mathcal{J}_n)$, the semilattice of idempotents of (the isomorphic copy of) the dual symmetric inverse monoid $\mathcal{J}_n \subseteq \mathcal{P}_n$; see [13]. In particular, $E(\mathcal{J}_n)$ is isomorphic to the semilattice (\mathfrak{Eq}_n, \vee) ; so $\bar{t}_\varepsilon \bar{t}_\eta = \bar{t}_{\varepsilon \vee \eta}$ for all $\varepsilon, \eta \in \mathfrak{Eq}_n$. By the discussion above, we see that $E(\mathcal{J}_n)$ is generated (as a monoid) by the set $\{\bar{t}_{ij} : 1 \leq i < j \leq n\}$. Theorem 2 of [11] says that $E(\mathcal{J}_n)$ has (monoid) presentation $\langle T : (\text{R3–R5}) \rangle$ via the map $t_{ij} \mapsto \bar{t}_{ij}$.

Proposition 3.1. *The semigroup $\mathcal{P}_n \setminus \mathcal{S}_n$ is generated by the set $\{\bar{e}_r : 1 \leq r \leq n\} \cup \{\bar{t}_{ij} : 1 \leq i < j \leq n\}$. In particular, the map $\phi : (E \cup T)^+ \rightarrow \mathcal{P}_n \setminus \mathcal{S}_n$ is surjective.*

Proof. Let $\alpha \in \mathcal{P}_n \setminus \mathcal{S}_n$, and write

$$\alpha = \left(\begin{array}{c|c|c|c|c} A_1 & \dots & A_r & C_1 & \dots & C_p \\ B_1 & \dots & B_r & D_1 & \dots & D_q \end{array} \right),$$

noting that $r = \text{rank}(\alpha) \leq n - 1$. For each $1 \leq i \leq r$, choose some $a_i \in A_i$ and $b_i \in B_i$. Then clearly $\alpha = \beta\gamma\delta$, where $\beta = \left(\begin{smallmatrix} A_1 & \dots & A_r & C_1 & \dots & C_p \\ A_1 & \dots & A_r & C_1 & \dots & C_p \end{smallmatrix} \right)$, $\gamma = \left[\begin{smallmatrix} a_1 & \dots & a_r \\ b_1 & \dots & b_r \end{smallmatrix} \right]$ and $\delta = \left(\begin{smallmatrix} B_1 & \dots & B_r & D_1 & \dots & D_q \\ B_1 & \dots & B_r & D_1 & \dots & D_q \end{smallmatrix} \right)$. Note that $\beta = \bar{t}_{\ker(\alpha)}$ and $\delta = \bar{t}_{\text{coker}(\alpha)}$, so that $\beta, \delta \in \langle \bar{t}_{ij} : 1 \leq i < j \leq n \rangle \cup \{1\}$, by the discussion before the statement of the proposition. By Theorem 2.3, $\gamma \in \mathcal{I}_n \setminus \mathcal{S}_n$ is a (non-empty) product of terms of the form $\bar{f}_{ij} = \bar{e}_i \bar{t}_{ij} \bar{e}_j$. \square

Remark 3.2. Proposition 3.1 was also proven in [9, Theorem 9], using a classical result of Howie on transformation semigroups [15]. Since $\{\bar{e}_r : 1 \leq r \leq n\} \subseteq E(\mathcal{I}_n)$ and $\{\bar{t}_{ij} : 1 \leq i < j \leq n\} \subseteq E(\mathcal{J}_n)$, it follows that $\langle E(\mathcal{P}_n) \rangle = \langle E(\mathcal{I}_n) \cup E(\mathcal{J}_n) \rangle = \{1\} \cup (\mathcal{P}_n \setminus \mathcal{S}_n)$.

Now that we know ϕ is surjective, it remains to show that $\ker \phi = \sim$, where \sim is the congruence on $(E \cup T)^+$ generated by the relations (R1–R10). For $w \in (E \cup T)^+$, we write $\bar{w} = w\phi$. Even though the empty word 1 does not belong to $(E \cup T)^+$, we will also write $\bar{1} = 1$ and $1 \sim 1$.

Lemma 3.3. *We have $\sim \subseteq \ker \phi$.*

Proof. This follows by a simple diagrammatic check that ϕ preserves the relations (R1–R10). We do this for (R7) in Figure 1, and leave the rest for the reader. \square

For $i, j \in \mathbf{n}$ with $i \neq j$, define the word $z_{ij} = e_i t_{ij} e_j \in (E \cup T)^+$, and write $Z = \{z_{ij} : i, j \in \mathbf{n}, i \neq j\}$ for the set of all such words. Note that $\bar{z}_{ij} = z_{ij}\phi = \bar{f}_{ij} \in \mathcal{I}_n$. The words z_{ij} will play a crucial role in what follows. We first prove a number of basic relations satisfied among them.

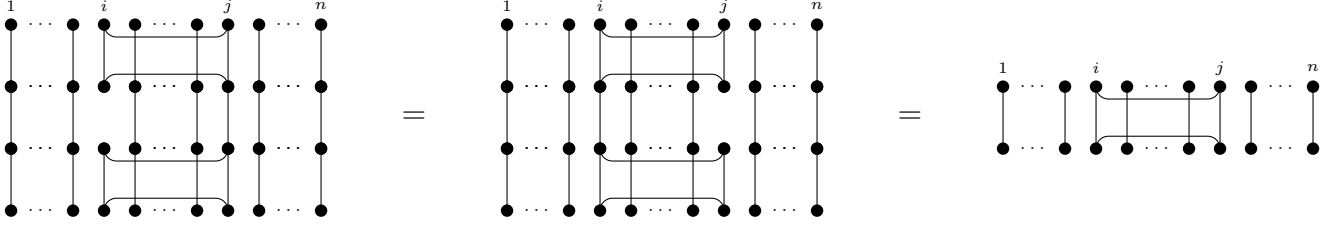


Figure 1: Diagrammatic proof of relation (R7): $\bar{t}_{ij}\bar{e}_i\bar{t}_{ij} = \bar{t}_{ij}\bar{e}_j\bar{t}_{ij} = \bar{t}_{ij}$.

Lemma 3.4. *Let $i, j, k \in \mathbf{n}$ be distinct. Then*

- (i) $e_i z_{ij} \sim z_{ij} \sim z_{ij} e_j$; (ii) $e_j z_{ij} \sim z_{ij} e_i \sim e_i e_j \sim z_{ij}^2 \sim z_{ji}^2$; (iii) $z_{ij} z_{ji} \sim e_i$; (iv) $e_k z_{ij} \sim z_{ij} e_k$.

Proof. Part (i) follows immediately from (R1). For (ii), note that $e_j z_{ij} = e_j e_i t_{ij} e_j \sim e_i e_j t_{ij} e_j \sim e_i e_j$, by (R2) and (R8). A similar calculation gives $z_{ij} e_i \sim e_i e_j$. Together with (R8), it also follows that $z_{ij}^2 = z_{ij} e_i t_{ij} e_j \sim e_i e_j t_{ij} e_j \sim e_i e_j$; similarly, $z_{ji}^2 \sim e_i e_j$, completing the proof of (ii). For (iii), we have $z_{ij} z_{ji} = e_i t_{ij} e_j e_j t_{ij} e_i \sim e_i t_{ij} e_j t_{ij} e_i \sim e_i t_{ij} e_i \sim e_i$, by (R1), (R7) and (R8). Finally, (iv) follows immediately from (R2) and (R6). \square

Recall that for $\alpha \in \mathcal{I}_n$ and $x \in \text{dom}(\alpha)$, we write $x\alpha$ for the (unique) element of $\text{codom}(\alpha)$ such that $\{x, (x\alpha)'\}$ is a block of α . Recall also that we are using symmetric notation for the $t_{ij} = t_{ji}$.

Lemma 3.5. *Let $i, j, k, l \in \mathbf{n}$ with $i \neq j$, $k \neq l$ and $k \notin \{i, j\}$. Then*

$$t_{ij} z_{kl} \sim z_{kl} t_{i\bar{z}_{kl,j}\bar{z}_{kl}} = \begin{cases} z_{kl} t_{ij} & \text{if } l \notin \{i, j\} \\ z_{kl} t_{kj} & \text{if } l = i \\ z_{kl} t_{ik} & \text{if } l = j. \end{cases}$$

Proof. The case in which $l \notin \{i, j\}$ follows immediately from (R4) and (R6). If $l = i$, then

$$t_{ij} z_{kl} = t_{ij} e_k t_{kl} e_l \sim e_k t_{ij} t_{kl} e_l = e_k t_{jl} t_{lk} e_l \sim e_k t_{lk} t_{kj} e_l \sim e_k t_{kl} e_l t_{kj} = z_{kl} t_{kj},$$

by (R5) and (R6). The $l = j$ case is similar. \square

Recall that $\langle Z \rangle$ denotes the subsemigroup of $(E \cup T)^+$ generated by Z .

Corollary 3.6. *If $w \in \langle Z \rangle$ and $i, j \in \text{dom}(\bar{w})$ with $i \neq j$, then $t_{ij} w \sim w t_{i\bar{w}, j\bar{w}}$.*

Proof. This follows immediately from Lemma 3.5, after writing $w = z_{k_1 l_1} \cdots z_{k_s l_s}$. \square

The next result shows that (modulo the relations (R1–R10)) the words z_{ij} satisfy the defining relations for $\mathcal{I}_n \setminus \mathcal{S}_n$ from Theorem 2.3.

Lemma 3.7. *For distinct $i, j, k, l \in \mathbf{n}$, we have*

$$z_{ij} z_{ji} z_{ij} \sim z_{ij} \tag{Z1}$$

$$z_{ij}^3 \sim z_{ij}^2 \sim z_{ji}^2 \tag{Z2}$$

$$z_{ij} z_{kl} \sim z_{kl} z_{ij} \tag{Z3}$$

$$z_{ij} z_{ji} \sim z_{ik} z_{ki} \tag{Z4}$$

$$z_{ij} z_{ik} \sim z_{jk} z_{ij} \sim z_{ik} z_{jk} \tag{Z5}$$

$$z_{ki} z_{ij} z_{jk} \sim z_{kj} z_{ji} z_{ik} \tag{Z6}$$

$$z_{ki} z_{ij} z_{jk} z_{kl} \sim z_{kl} z_{li} z_{ij} z_{jl}. \tag{Z7}$$

Proof. For (Z1), we have $z_{ij}z_{ji}z_{ij} \sim e_i z_{ij} \sim z_{ij}$, by Lemma 3.4(iii) and (i). For (Z2), note that $z_{ij}^2 \sim z_{ji}^2 \sim e_i e_j$ by Lemma 3.4(ii); it also follows that $z_{ij}^3 \sim e_i e_j z_{ij} \sim e_i e_i e_j \sim e_i e_j$ by Lemma 3.4(ii) and (R1), completing the proof of (Z2). Relation (Z3) follows immediately from (R2), (R4) and (R6). Relation (Z4) follows immediately from Lemma 3.4(iii). For (Z5), first note that $z_{ij}z_{ik} \sim z_{ij}e_i z_{ik} \sim e_i e_j z_{ik} \sim e_j z_{ik}$, by Lemma 3.4(i) and (ii), and (R2). We also have

$$\begin{aligned}
z_{jk}z_{ij} &= e_j t_{jk} e_k e_i t_{ij} e_j \sim e_i e_j t_{jk} t_{ij} e_j e_k && \text{by (R2) and (R6)} \\
&= e_i e_j t_{kj} t_{ji} e_j e_k \\
&\sim e_i e_j t_{ji} t_{ik} e_j e_k && \text{by (R5)} \\
&\sim e_i e_j t_{ji} e_j t_{ik} e_k && \text{by (R6)} \\
&\sim e_i e_j t_{ik} e_k && \text{by (R8)} \\
&\sim e_j e_i t_{ik} e_k = e_j z_{ik} && \text{by (R2).}
\end{aligned}$$

By Lemma 3.4(i), (ii) and (iv), $z_{ik}z_{jk} \sim z_{ik}e_k z_{jk} \sim z_{ik}e_j e_k \sim e_j z_{ik}e_k \sim e_j z_{ik}$, completing the proof of (Z5). For (Z6), we have

$$z_{ki}z_{ij}z_{jk} = e_k t_{ki} e_i e_j t_{ij} e_j t_{jk} e_k \sim e_k t_{ki} e_i t_{ij} e_j t_{jk} e_k \sim e_k t_{kj} e_j t_{ji} e_i t_{ik} e_k \sim e_k t_{kj} e_j e_j t_{ji} e_i t_{ik} e_k = z_{kj}z_{ji}z_{ik},$$

by (R1) and (R9). Finally, for (Z7), first observe that by a similar calculation to that just carried out, (R10) gives $z_{ki}z_{ij}z_{jl}z_{lk} \sim z_{kl}z_{li}z_{ij}z_{jk}$. We then have

$$\begin{aligned}
z_{ki}z_{ij}z_{jk}z_{kl} &\sim z_{ki}z_{ij}z_{ji}z_{ij}z_{jk}z_{kl} && \text{by (Z1)} \\
&\sim z_{ki}z_{ij}z_{jl}z_{lj}z_{jk}z_{kl} && \text{by (Z4)} \\
&\sim z_{ki}z_{ij}z_{jl}z_{lk}z_{kj}z_{jl} && \text{by (Z6)} \\
&\sim z_{kl}z_{li}z_{ij}z_{jk}z_{kj}z_{jl} && \text{by the observation} \\
&\sim z_{kl}z_{li}z_{ij}z_{jl}z_{lj}z_{jl} && \text{by (Z4)} \\
&\sim z_{kl}z_{li}z_{ij}z_{jl} && \text{by (Z1).}
\end{aligned}$$

This completes the proof. □

Corollary 3.8. *If $u, v \in \langle Z \rangle$, then $\bar{u} = \bar{v} \Rightarrow u \sim v$.*

Proof. Write $u = z_{i_1 j_1} \cdots z_{i_s j_s}$ and $v = z_{k_1 l_1} \cdots z_{k_t l_t}$, and suppose $\bar{u} = \bar{v}$. Then

$$\bar{f}_{i_1 j_1} \cdots \bar{f}_{i_s j_s} = \bar{z}_{i_1 j_1} \cdots \bar{z}_{i_s j_s} = \bar{u} = \bar{v} = \bar{z}_{k_1 l_1} \cdots \bar{z}_{k_t l_t} = \bar{f}_{k_1 l_1} \cdots \bar{f}_{k_t l_t}.$$

By Theorem 2.3, the word $f_{i_1 j_1} \cdots f_{i_s j_s}$ may be transformed into $f_{k_1 l_1} \cdots f_{k_t l_t}$ using relations (F1–F7). By Lemma 3.7, this transformation leads to a transformation of u into v using (Z1–Z7). □

The next result follows from [11, Theorem 2], which (as noted above) states that $E(\mathcal{J}_n) \cong (\mathfrak{E}_{\mathbf{q}_n}, \vee)$ has monoid presentation $\langle T : (\text{R3–R5}) \rangle$ via $t_{ij} \mapsto \bar{t}_{ij}$.

Lemma 3.9. *If $u, v \in T^*$, then $\bar{u} = \bar{v} \Rightarrow u \sim v$.* □

In order to complete the proof of Theorem 2.1, we aim to show that any word over $E \cup T$ may be rewritten (using the relations) to take on a very specific form; see Proposition 3.14, the proof of which requires the next three intermediate lemmas.

Lemma 3.10. *If $w \in (E \cup T)^+$, then $w \sim w_1 w_2 w_3$ for some $w_1, w_3 \in T^*$ and $w_2 \in \langle Z \rangle$.*

Proof. We prove this by induction on $\ell(w)$, the length of the word w . Suppose first that $\ell(w) = 1$. If $w = e_i$ for some i , then $w \sim z_{ij}z_{ji}$ for any $j \in \mathbf{n} \setminus \{i\}$, by Lemma 3.4(iii), and we are done (with $w_1 = w_3 = 1$ and $w_2 = z_{ij}z_{ji}$). If $w = t_{ij}$ for some $1 \leq i < j \leq n$, then $w = t_{ij} \sim t_{ij}e_it_{ij} \sim t_{ij}z_{ij}z_{ji}t_{ij}$, by (R7) and Lemma 3.4(iii), and we are done (with $w_1 = w_3 = t_{ij}$ and $w_2 = z_{ij}z_{ji}$).

Now suppose $\ell(w) \geq 2$, and write $w = ux$, where $u \in (E \cup T)^+$ and $x \in E \cup T$. By an inductive hypothesis, $u \sim u_1u_2u_3$ for some $u_1, u_3 \in T^*$ and $u_2 \in \langle Z \rangle$. If $x \in T$, then $w \sim u_1u_2u_3x$, and we are done (with $w_1 = u_1$, $w_2 = u_2$ and $w_3 = u_3x$). So suppose $x = e_i \in E$, and write $u_3 = t_{k_1l_1} \cdots t_{k_sl_s}$.

Case 1. If $i \notin \{k_1, \dots, k_s, l_1, \dots, l_s\}$, then $u_3e_i \sim e_iu_3 \sim z_{ij}z_{ji}u_3$ for any $j \in \mathbf{n} \setminus \{i\}$, by (R6) and Lemma 3.4(iii), so $w \sim u_1u_2z_{ij}z_{ji}u_3$, and we are done (with $w_1 = u_1$, $w_2 = u_2z_{ij}z_{ji}$ and $w_3 = u_3$).

Case 2. Now suppose $i \in \{k_1, \dots, k_s, l_1, \dots, l_s\}$. By (R3), (R4), and the symmetrical notation for the $t_{kl} = t_{lk}$, we may assume that in fact $u_3 = t_{il_1} \cdots t_{il_r} \cdot t_{k_{r+1}l_{r+1}} \cdots t_{k_sl_s}$ with $r \geq 1$ and $l_1 < \cdots < l_r$. Now $t_{il_1} \cdots t_{il_r} \sim t_{il_1}t_{l_1l_2} \cdots t_{l_{r-1}l_r}$, by Lemma 3.9. It follows that

$$w \sim u_1u_2t_{il_1}t_{l_1l_2} \cdots t_{l_{r-1}l_r}t_{k_{r+1}l_{r+1}} \cdots t_{k_sl_s}e_i \sim u_1 \cdot u_2t_{il_1}e_i \cdot u_4,$$

where $u_4 = t_{l_1l_2} \cdots t_{l_{r-1}l_r}t_{k_{r+1}l_{r+1}} \cdots t_{k_sl_s}$. For simplicity, we will write $j = l_1$. Since $u_1, u_4 \in T^*$, the proof will be complete if we can show that $u_2t_{ij}e_i \sim v_1v_2v_3$ for some $v_1, v_3 \in T^*$ and $v_2 \in \langle Z \rangle$.

Subcase 2.1. Suppose first that $i, j \in \text{codom}(\bar{u}_2)$, and let $k, l \in \text{dom}(\bar{u}_2)$ be such that $k\bar{u}_2 = i$ and $l\bar{u}_2 = j$. Then $t_{kl}u_2 \sim u_2t_{ij}$, by Corollary 3.6, whence $u_2t_{ij}e_i \sim t_{kl}u_2e_i \sim t_{kl}u_2z_{ij}z_{ji}$, and we are done (with $v_1 = t_{kl}$, $v_2 = u_2z_{ij}z_{ji}$ and $v_3 = 1$).

Subcase 2.2. Next suppose $i \notin \text{codom}(\bar{u}_2)$. Then $u_2 \sim u_2z_{ij}z_{ji} \sim u_2e_i$, by Corollary 3.8 and Lemma 3.4(iii). But then, together with (R8), it follows that $u_2t_{ij}e_i \sim u_2e_it_{ij}e_i \sim u_2e_i \sim u_2$, and we are done (with $v_1 = v_3 = 1$ and $v_2 = u_2$).

Subcase 2.3. Finally, suppose $j \notin \text{codom}(\bar{u}_2)$. Then $u_2 \sim u_2e_j$, as in the previous case, giving $u_2t_{ij}e_i \sim u_2e_jt_{ij}e_i = u_2z_{ji}$, and we are done (with $v_1 = v_3 = 1$ and $v_2 = u_2z_{ij}$). \square

To improve Lemma 3.10, we first define some words over T . Consider a subset $A \subseteq \mathbf{n}$. If $|A| \leq 1$, then put $t_A = 1$. Otherwise, write $A = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$, and define $t_A = t_{i_1i_2}t_{i_2i_3} \cdots t_{i_{k-1}i_k}$. For an equivalence $\varepsilon \in \mathfrak{E}\mathbf{q}_n$ with equivalence classes A_1, \dots, A_r with $\min(A_1) < \cdots < \min(A_r)$, define $t_\varepsilon = t_{A_1} \cdots t_{A_r}$. Note that if $w \in T^*$ is such that $\bar{w} = \bar{t}_\varepsilon = \begin{pmatrix} A_1 & \cdots & A_r \\ A_1 & \cdots & A_r \end{pmatrix}$, then $w \sim t_\varepsilon$, by Lemma 3.9. For the proof of the next result, if $\varepsilon \in \mathfrak{E}\mathbf{q}_n$, we write \mathbf{n}/ε for the set of all ε -classes.

Lemma 3.11. *Let $w \in (E \cup T)^+$, and put $\varepsilon = \ker(\bar{w})$ and $\eta = \text{coker}(\bar{w})$. Then $w \sim t_\varepsilon u t_\eta$ for some $u \in \langle Z \rangle$.*

Proof. By Lemmas 3.9 and 3.10, the set $\{(\lambda, u, \rho) \in \mathfrak{E}\mathbf{q}_n \times \langle Z \rangle \times \mathfrak{E}\mathbf{q}_n : w \sim t_\lambda u t_\rho\}$ is non-empty. Choose an element (λ, u, ρ) from this set such that $k = |\mathbf{n}/\lambda| + |\mathbf{n}/\rho|$ is minimal. Suppose the λ -classes and ρ -classes are A_1, \dots, A_p and B_1, \dots, B_q , respectively (so $p = |\mathbf{n}/\lambda|$ and $q = |\mathbf{n}/\rho|$). Note that $\lambda = \ker(\bar{t}_\lambda) \subseteq \ker(\bar{t}_\lambda \bar{u} \bar{t}_\rho) = \ker(\bar{w}) = \varepsilon$ and, similarly, $\rho \subseteq \eta$. In particular, each ε -class is a union of (one or more) λ -classes, with a similar statement holding for η -classes and ρ -classes. Let the ε -classes and η -classes be C_1, \dots, C_s and D_1, \dots, D_t , respectively (so $s = |\mathbf{n}/\varepsilon|$ and $t = |\mathbf{n}/\eta|$), noting that $s \leq p$ and $t \leq q$. In particular, $k = p + q \geq s + t$. If $k = s + t$, then $p = s$ and $q = t$, so that $\lambda = \varepsilon$ and $\rho = \eta$, and the proof would be complete. So suppose instead that $k > s + t$. Note that one of the following statements must be true:

- (i) there exist $a, b \in \text{dom}(\bar{u})$ such that $(a, b) \in \lambda$ and $(a\bar{u}, b\bar{u}) \notin \rho$; or
- (ii) there exist $a, b \in \text{dom}(\bar{u})$ such that $(a, b) \notin \lambda$ and $(a\bar{u}, b\bar{u}) \in \rho$.

Indeed, if (i) and (ii) were both false, then we would have $\lambda = \ker(\bar{t}_\lambda \bar{u} \bar{t}_\rho) = \ker(\bar{w}) = \varepsilon$ and, similarly, $\rho = \eta$, contradicting the assumption that $k > s + t$. Suppose (i) is true (the other case being similar), and write $c = a\bar{u}$ and $d = b\bar{u}$. Relabelling if necessary, we may assume that $a, b \in A_1$, $c \in B_1$ and $d \in B_2$. Since $(a, b) \in \lambda$, Lemma 3.9 gives $t_\lambda \sim t_\lambda t_{ab}$. Let $\varepsilon_{cd} \in \mathfrak{E}\mathfrak{q}_n$ be the equivalence whose only non-singleton equivalence class is $\{c, d\}$, and let $\kappa = \varepsilon_{cd} \vee \rho$. Then Lemma 3.9 also gives $t_{cd} t_\rho = t_{\varepsilon_{cd}} t_\rho \sim t_\kappa$. Together with Corollary 3.6, it then follows that $w \sim t_\lambda t_{ab} u t_\rho \sim t_\lambda u t_{cd} t_\rho \sim t_\lambda u t_\kappa$. But κ has $q - 1$ equivalence classes (i.e., $B_1 \cup B_2, B_3, \dots, B_q$), so $|\mathbf{n}/\lambda| + |\mathbf{n}/\kappa| = k - 1$, contradicting the minimality of k . \square

Remark 3.12. The proof of Lemma 3.11 is set out as a *reductio* merely for convenience. It is easily adapted to become constructive. (The same is true of Lemma 3.13 and Proposition 3.14).

Lemma 3.13. *Let $w \in (E \cup T)^+$, and put $\varepsilon = \ker(\bar{w})$ and $\eta = \text{coker}(\bar{w})$. Then $w \sim t_\varepsilon u t_\eta$ for some $u \in \langle Z \rangle$ with $\text{rank}(\bar{u}) = \text{rank}(\bar{w})$.*

Proof. Put $r = \text{rank}(\bar{w})$. By Lemma 3.11, the set $\{u \in \langle Z \rangle : w \sim t_\varepsilon u t_\eta\}$ is non-empty. Choose an element u from this set with $\text{rank}(\bar{u})$ minimal. If $\text{rank}(\bar{u}) = r$, then we are done, so suppose otherwise. Since $r = \text{rank}(\bar{w}) = \text{rank}(\bar{t}_\varepsilon \bar{u} \bar{t}_\eta) \leq \text{rank}(\bar{u})$, it follows that $\text{rank}(\bar{u}) > r$. Suppose the ε -classes and η -classes are A_1, \dots, A_p and B_1, \dots, B_q , respectively, and that the transversal blocks of \bar{w} are $A_1 \cup B'_1, \dots, A_r \cup B'_r$. Note that $\text{dom}(\bar{u}) \subseteq A_1 \cup \dots \cup A_r$ and $\text{codom}(\bar{u}) \subseteq B_1 \cup \dots \cup B_r$. Since $\text{rank}(\bar{u}) > r$, we may assume (relabelling if necessary) that $|A_1 \cap \text{dom}(\bar{u})| \geq 2$. Let $i, j \in A_1 \cap \text{dom}(\bar{u})$ with $i \neq j$, and put $k = i\bar{u}$ and $l = j\bar{u}$, noting that $k, l \in B_1$. It follows from Lemma 3.9 and Corollary 3.6 that $t_\varepsilon \sim t_\varepsilon t_{ij}$, $t_\eta \sim t_{kl} t_\eta$, and $t_{ij} u \sim u t_{kl}$. Together with (R7) and Lemma 3.4(iii), we then obtain $w \sim t_\varepsilon u t_\eta \sim t_\varepsilon t_{ij} u t_\eta \sim t_\varepsilon t_{ij} e_i t_{ij} u t_\eta \sim t_\varepsilon t_{ij} e_i u t_{kl} t_\eta \sim t_\varepsilon (z_{ij} z_{ji} u) t_\eta$. But $\text{dom}(\bar{z}_{ij} \bar{z}_{ji} \bar{u}) = \text{dom}(\bar{u}) \setminus \{i\}$, so that $\text{rank}(\bar{z}_{ij} \bar{z}_{ji} \bar{u}) = \text{rank}(\bar{u}) - 1$, contradicting the minimality of $\text{rank}(\bar{u})$. \square

The next result gives a set of *normal forms* for words over $E \cup T$, and is the final ingredient in the proof of Theorem 2.1. For each $\alpha \in \mathcal{I}_n \setminus \mathcal{S}_n$, fix some $z_\alpha \in \langle Z \rangle$ such that $\bar{z}_\alpha = z_\alpha \phi = \alpha$.

Proposition 3.14. *Let $w \in (E \cup T)^+$, and write*

$$\bar{w} = \left(\begin{array}{c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_p \\ \hline B_1 & \cdots & B_r & D_1 & \cdots & D_q \end{array} \right), \quad \varepsilon = \ker(\bar{w}), \quad \eta = \text{coker}(\bar{w}), \quad \alpha = \left[\begin{array}{c|c|c} a_1 & \cdots & a_r \\ \hline b_1 & \cdots & b_r \end{array} \right],$$

where $a_i = \min(A_i)$ and $b_i = \min(B_i)$ for each i . Then $w \sim t_\varepsilon z_\alpha t_\eta$.

Proof. By Lemma 3.13, the set $\{u \in \langle Z \rangle : \text{rank}(\bar{u}) = r, w \sim t_\varepsilon u t_\eta\}$ is non-empty. Choose some u from this set with $k = |\text{dom}(\bar{u}) \cap \text{dom}(\alpha)| + |\text{codom}(\bar{u}) \cap \text{codom}(\alpha)|$ maximal. We claim that $k = 2r$. Indeed, suppose to the contrary that $k < 2r$, and write $\bar{u} = \left[\begin{array}{c|c|c} c_1 & \cdots & c_r \\ \hline d_1 & \cdots & d_r \end{array} \right]$, where $c_i \in A_i$ and $d_i \in B_i$ for each i . Since $k < 2r$, it follows that $c_i \neq a_i$ or $d_i \neq b_i$ for some i . We assume the former is the case (the latter is treated in similar fashion). Relabelling if necessary, we may assume that $i = 1$; for simplicity, we will write $a = a_1$ and $c = c_1$. Since $a \notin \text{dom}(\bar{u})$, we have $u \sim z_{ac} z_{ca} u = e_a t_{ac} e_c z_{ca} u \sim e_a t_{ac} z_{ca} u$, by Corollary 3.8 and Lemma 3.4(i). Since $a, c \in A_1$, Lemma 3.9 gives $t_\varepsilon \sim t_\varepsilon t_{ac}$. Together with (R7), it follows that $w \sim t_\varepsilon u t_\eta \sim t_\varepsilon t_{ac} (e_a t_{ac} z_{ca} u) t_\eta \sim t_\varepsilon t_{ac} z_{ca} u t_\eta \sim t_\varepsilon (z_{ca} u) t_\eta$. Note that $\bar{z}_{ca} \bar{u} = \left[\begin{array}{c|c|c} a_1 & c_2 & \cdots & c_r \\ \hline d_1 & d_2 & \cdots & d_r \end{array} \right]$. But $|\text{dom}(\bar{z}_{ca} \bar{u}) \cap \text{dom}(\alpha)| + |\text{codom}(\bar{z}_{ca} \bar{u}) \cap \text{codom}(\alpha)| = k + 1$, contradicting the maximality of k . This completes the proof of the claim. It follows that $\bar{u} = \alpha$, so Corollary 3.8 gives $u \sim z_\alpha$. \square

We now have all we need to complete the proof of the first main result.

Proof of Theorem 2.1. It remains to check that $\ker \phi \subseteq \sim$, so suppose $w, v \in (E \cup T)^+$ are such that $\bar{w} = \bar{v}$. Then $w \sim t_\varepsilon z_\alpha t_\eta$, in the notation of Proposition 3.14. Since $\bar{v} = \bar{w}$, we also have $v \sim t_\varepsilon z_\alpha t_\eta$, so that $w \sim v$. \square

4 Presentation for \mathcal{P}_n

Again, the proof of Theorem 2.2 involves two steps: showing that the map $\Phi : (S \cup \{e, t\})^* \rightarrow \mathcal{P}_n$ is an epimorphism; and showing that $\ker \Phi$ is generated by relations (R11–R21).

Write \approx for the congruence on $(S \cup \{e, t\})^*$ generated by (R11–R21). Without causing confusion, we will write $\bar{w} = w\Phi$ for any $w \in (S \cup \{e, t\})^*$. For $w = s_{i_1} \cdots s_{i_k} \in S^*$, we will write $w^{-1} = s_{i_k} \cdots s_{i_1}$, noting that $ww^{-1} \approx w^{-1}w \approx 1$, by (R11). Note also that the symmetric group $\mathcal{S}_n \subseteq \mathcal{P}_n$ has monoid presentation $\langle S : (\text{R11–R13}) \rangle$ via $s_i \mapsto \bar{s}_i$; see [23].

For $1 \leq r \leq n$ and $1 \leq i < j \leq n$, define the words

$$c_r = s_1 \cdots s_{r-1}, \quad \epsilon_r = c_r^{-1} e c_r, \quad \tau_{ij} = \tau_{ji} = c_i^{-1} c_j^{-1} t c_j c_i.$$

One may easily check diagrammatically that $\bar{\epsilon}_r = \epsilon_r \Phi = \bar{\epsilon}_r$ and $\bar{\tau}_{ij} = \tau_{ij} \Phi = \bar{\tau}_{ij}$. In particular, $\text{im } \Phi$ contains $\mathcal{P}_n \setminus \mathcal{S}_n$, by Proposition 3.1. Since also $S\Phi = \{\bar{s}_r : 1 \leq r \leq n-1\}$ generates \mathcal{S}_n , it follows that Φ is surjective. It is easy to check that each of relations (R11–R21) are preserved by Φ . So, to prove Theorem 2.2, it remains to check that $\ker \Phi \subseteq \approx$, and the rest of this section is devoted to that task. We begin with some simple properties of the words ϵ_r and τ_{ij} .

Lemma 4.1. *If $1 \leq r \leq n$ and $1 \leq k \leq n-1$, then $s_k \epsilon_r s_k \approx \epsilon_r \bar{s}_k$.*

Proof. We must show that

$$s_k \epsilon_r s_k \approx \begin{cases} \epsilon_{r-1} & \text{if } k = r-1 \\ \epsilon_{r+1} & \text{if } k = r \\ \epsilon_r & \text{otherwise.} \end{cases}$$

These follows quickly from (R11), (R16), and the easily checked facts that

$$c_r s_k \approx \begin{cases} s_{k+1} c_r & \text{if } k \leq r-2 \\ c_{r-1} & \text{if } k = r-1 \\ c_{r+1} & \text{if } k = r \\ s_k c_r & \text{if } k \geq r+1 \end{cases} \quad \text{and} \quad s_k c_r^{-1} \approx \begin{cases} c_r^{-1} s_{k+1} & \text{if } k \leq r-2 \\ c_{r-1}^{-1} & \text{if } k = r-1 \\ c_{r+1}^{-1} & \text{if } k = r \\ c_r^{-1} s_k & \text{if } k \geq r+1. \end{cases}$$

For example, if $k = r-1$, then $s_k \epsilon_r s_k = s_{r-1} c_r^{-1} e c_r s_{r-1} \approx c_{r-1}^{-1} e c_{r-1} = \epsilon_{r-1}$, while if $k \leq r-2$, then $s_k \epsilon_r s_k = s_k c_r^{-1} e c_r s_k \approx c_r^{-1} s_{k+1} e s_{k+1} c_r \approx c_r^{-1} s_{k+1} s_{k+1} e c_r \approx c_r^{-1} e c_r = \epsilon_r$. \square

Lemma 4.2 (cf. [11, p322]). *If $1 \leq i < j \leq n$ and $1 \leq k \leq n-1$, then $s_k \tau_{ij} s_k \approx \tau_{i\bar{s}_k, j\bar{s}_k}$.*

Proof. This follows by a similar proof to that of Lemma 4.1, using the same relations satisfied by the c_r, s_k . For example, if $k = j$, then

$$s_k \tau_{ij} s_k = s_j c_i^{-1} c_j^{-1} t c_j c_i s_j \approx c_i^{-1} s_j c_j^{-1} t c_j s_j c_i \approx c_i^{-1} c_{j+1}^{-1} t c_{j+1} c_i = \tau_{i, j+1}. \quad \square$$

Corollary 4.3. *If $1 \leq r \leq n$, $1 \leq i < j \leq n$, and $w \in S^*$, then $w^{-1} \epsilon_r w \approx \epsilon_r \bar{w}$ and $w^{-1} \tau_{ij} w \approx \tau_{i\bar{w}, j\bar{w}}$.*

Proof. This follows from Lemmas 4.1 and 4.2 and a simple induction on the length of w . \square

We now aim to link the presentations $\langle E \cup T : (\text{R1–R10}) \rangle$ and $\langle S \cup \{e, t\} : (\text{R11–R21}) \rangle$ in a certain sense. With this in mind, we define a map $\psi : (E \cup T)^+ \rightarrow (S \cup \{e, t\})^*$ by $e_r \psi = \epsilon_r$ and $t_{ij} \psi = \tau_{ij}$. Note that $\bar{u} = \overline{u\psi}$ (i.e., $u\phi = u\psi\Phi$) for all $u \in (E \cup T)^+$. Recall that \sim is the congruence on $(E \cup T)^+$ generated by relations (R1–R10).

Lemma 4.4. *If $u, v \in (E \cup T)^+$, then $u \sim v \Rightarrow u\psi \approx v\psi$.*

Proof. We just need to check this for each relation $u = v$ from (R1–R10). Relations (R1) and (R3) follow immediately from (R11), (R14), (R15). For (R2), first note that $\epsilon_1\epsilon_2 = es_1es_1 \approx s_1es_1e = \epsilon_2\epsilon_1$, by (R18). Together with Corollary 4.3, it follows that for any $w \in S^*$ with $1\overline{w} = i$ and $2\overline{w} = j$, $\epsilon_i\epsilon_j \approx w^{-1}\epsilon_1ww^{-1}\epsilon_2w \approx w^{-1}\epsilon_1\epsilon_2w \approx w^{-1}\epsilon_2\epsilon_1w \approx w^{-1}\epsilon_2ww^{-1}\epsilon_1w \approx \epsilon_j\epsilon_i$. The other relations are all treated in similar fashion; in each case, we use Corollary 4.3 to reduce the calculation to a fixed set of values of the subscripts. For example, for (R9), taking $(i, j, k) = (1, 2, 3)$,

$$\begin{aligned}
\epsilon_3\tau_{31}\epsilon_1\tau_{12}\epsilon_2\tau_{23}\epsilon_3 &= (s_2s_1es_1s_2)(s_2s_1ts_1s_2)et(s_1es_1)(s_1s_2s_1ts_1s_2s_1)(s_2s_1es_1s_2) \\
&\approx s_2s_1ets_2etes_2ts_2s_1s_2s_1es_1s_2 && \text{by (R11) and (R15)} \\
&\approx s_2s_1ets_2es_2ts_2s_2s_1s_2es_1s_2 && \text{by (R13) and (R14)} \\
&\approx s_2s_1etes_2s_2ts_1s_2es_1s_2 && \text{by (R11) and (R16)} \\
&\approx s_2s_1etes_2s_1s_2 && \text{by (R11), (R14), (R15) and (R16)} \\
&\approx s_2s_1es_2s_1s_2 && \text{by (R14),}
\end{aligned}$$

with a similar calculation giving $\epsilon_3\tau_{32}\epsilon_2\tau_{21}\epsilon_1\tau_{13}\epsilon_3 \approx s_2s_1es_2s_1s_2$. \square

Lemma 4.5. *If $1 \leq r \leq n$ and $1 \leq k \leq n - 1$, then $\epsilon_r s_k$ and $s_k \epsilon_r$ are both \approx -equivalent to elements of $\text{im } \psi$.*

Proof. Since $s_k \epsilon_r \approx s_k \epsilon_r s_k s_k \approx \epsilon_r \overline{s_k} s_k$, by (R11) and Lemma 4.1, it suffices to show that

$$\epsilon_r s_k \approx \begin{cases} \epsilon_r \tau_{r,r-1} \epsilon_{r-1} & \text{if } k = r - 1 \\ \epsilon_r \tau_{r,r+1} \epsilon_{r+1} & \text{if } k = r \\ \epsilon_r \tau_{rk} \epsilon_k \tau_{k,k+1} \epsilon_{k+1} \tau_{k+1,r} \epsilon_r & \text{otherwise.} \end{cases}$$

We just treat the case in which $k \notin \{r - 1, r\}$, the others being easier. As above, we may assume that $k = 1$ and $r = 3$. Here we have $\epsilon_3\tau_{31}\epsilon_1\tau_{12}\epsilon_2\tau_{23}\epsilon_3 \approx s_2s_1es_2s_1s_2 \approx s_2s_1es_1s_2s_1 = \epsilon_3s_1$, by (R13) and the calculation from the proof of Lemma 4.4. \square

Lemma 4.6. *If $1 \leq i < j \leq n$ and $1 \leq k \leq n - 1$, then $\tau_{ij} s_k$ and $s_k \tau_{ij}$ are both \approx -equivalent to elements of $\text{im } \psi$.*

Proof. Again, it suffices to do this for just $\tau_{ij} s_k$. By (R11) and Lemmas 4.2 and 4.4, we have $\tau_{ij} s_k \approx \tau_{ij} \epsilon_i \tau_{ij} s_k \approx \tau_{ij} \epsilon_i s_k s_k \tau_{ij} s_k \approx \tau_{ij} (\epsilon_i s_k) \tau_{i\overline{s_k}, j\overline{s_k}}$, and the result now follows from Lemma 4.5. \square

Lemma 4.7. *If $w \in (S \cup \{e, t\})^* \setminus S^*$, then w is \approx -equivalent to an element of $\text{im } \psi$.*

Proof. Put $\Sigma = (E \cup T)\psi = \{\epsilon_r : 1 \leq r \leq n\} \cup \{\tau_{ij} : 1 \leq i < j \leq n\}$, noting that $\text{im } \psi = \langle \Sigma \rangle$. Since $e = \epsilon_1 \in \Sigma$ and $t \approx s_1ts_1 = \tau_{12} \in \Sigma$, it suffices to show that every element of $\langle \Sigma \cup S \rangle \setminus S^*$ is \approx -equivalent to an element of $\langle \Sigma \rangle$. With this in mind, let $w \in \langle \Sigma \cup S \rangle \setminus S^*$, and write $w = x_1 \cdots x_k$, where $x_1, \dots, x_k \in \Sigma \cup S$. Denote by l the number of factors x_i that belong to S . We proceed by induction on l . If $l = 0$, then we already have $w \in \langle \Sigma \rangle$, so suppose $l \geq 1$. Since $w \notin S^*$, there exists $1 \leq i \leq k - 1$ such that either (i) $x_i \in S$ and $x_{i+1} \in \Sigma$, or (ii) $x_i \in \Sigma$ and $x_{i+1} \in S$. In either case, Lemmas 4.5 and 4.6 tell us that $x_i x_{i+1} \approx u$ for some $u \in \text{im } \psi = \langle \Sigma \rangle$. But then $w \approx (x_1 \cdots x_{i-1})u(x_{i+2} \cdots x_k)$, and we are done, after applying an induction hypothesis (noting that $(x_1 \cdots x_{i-1})u(x_{i+2} \cdots x_k)$ has $l - 1$ factors from S). \square

We may now prove the second main result.

Proof of Theorem 2.2. It remains to show that $\ker \Phi \subseteq \approx$, so suppose $w_1, w_2 \in (S \cup \{e, t\})^*$ are such that $\bar{w}_1 = \bar{w}_2$. If $\bar{w}_1 \in \mathcal{S}_n$, then $w_1, w_2 \in S^*$, so $w_1 \approx w_2$, using only relations (R11–R13). So suppose $\bar{w}_1 \notin \mathcal{S}_n$. It follows that $w_1, w_2 \in (S \cup \{e, t\})^* \setminus S^*$. So, by Lemma 4.7, $w_1 \approx u_1\psi$ and $w_2 \approx u_2\psi$ for some $u_1, u_2 \in (E \cup T)^+$. We then have $\bar{u}_1 = \bar{u}_1\psi = \bar{w}_1 = \bar{w}_2 = \bar{u}_2\psi = \bar{u}_2$, so that $u_1 \sim u_2$, by Theorem 2.1. Lemma 4.4 then gives $u_1\psi \approx u_2\psi$, so that $w_1 \approx w_2$. \square

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